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On Decomposing c-Valued Division Rings

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INTRODUCTION

In earlier papers [C_1, C_2] a class of valuations on division algebras D with involution $*$ arose naturally in connection with the study of orderings on D. These valuations are called c-valuations (see Section 1 below for the definition). We consider here the question whether every c-valued division algebra D finite-dimensional over its center must be isomorphic to a tensor product of quaternion algebras. Such a decomposition would be highly desirable, since the c-valuations on quaternion algebras are quite well understood. This possibility is suggested by the information obtained in [C_2] about the value group and residue division algebra of a c-valuation on D, which make D "look like" a product of quaternion algebras. All the assorted finite-dimensional c-valued division algebras constructed in [C_2] are tensor products of quaternion algebras. Our chief result here (Theorem 2.1) is that the answer to our question is yes if the valuation on the center of D is Henselian. Indeed, we show in this case that D decomposes into quaternion algebras invariant under the involution. We deduce from this, using a theorem of Morandi, that any finite-dimensional c-valued division algebra D has an immediate extension, also c-valued, which decomposes into a product of quaternion algebras.

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However, we will also show that the original question has a negative answer by constructing in Section 4 examples of c-valued division algebras which do not decompose into quaternion algebras.

These results depend on an analysis of the valuation-theoretic properties of c-valuations, which is given in Section 1, and also on further basic constructions of c-valued division algebras, given in Section 3.

1. Basic Properties of c-Valued Division Algebras

We now set up basic notation and recall the general properties of valued division algebras and those specific to c-valued division algebras which will be needed below. In all that follows D will denote a division ring and F will be its center $Z(D)$. *We will assume throughout that the dimension* $[D:F]$ *is finite.*

Let Γ be a totally ordered abelian group, which we adjoin ∞ with $\gamma < \infty$ for all $\gamma \in \Gamma$. A *valuation* on the division ring D *with value group* Γ is an epimorphism $v: D \to \Gamma \cup \infty$ satisfying, for all $a, b \in D$,

(i) $v(a+b) \geqslant \min(v(a), v(b))$;

(ii) $v(ab) = v(a) + v(b)$;

(iii) $v(a) = \infty$ if and only if $a = 0$.

Let $V = \{a \in D \mid v(a) \geqslant 0\}$ be the valuation ring of v, and $M = \{a \in D \mid v(a) > 0\}$, the unique maximal left ideal (and maximal right ideal) of V. Let $U = V - M$, the group of v-units in D. Let $\bar{D} = V/M$, the residue division ring of v on D. For $a \in V$, we write $\bar{a}$ for the image of a in $\bar{D}$. When there is need to specify that the valuation is on D, we write Γ_D, V_D, M_D, U_D, or if necessary when there is more than one valuation, $\Gamma_{D,v}$, $V_{D,v}$, $M_{D,v}$, $U_{D,v}$, $\bar{D}_v$.

Let $D^{\boldsymbol{\cdot}} = D - \{0\}$, the multiplicative group of D. Observe that the group of 1-units of v, $1+M$, is a normal subgroup of $D^{\boldsymbol{\cdot}}$. We will work frequently with congruence modulo $1+M$. Note that for $a, b \in D^{\boldsymbol{\cdot}}$, $a \equiv b \pmod{1+M}$ if and only if $b = a + c$ with $v(c) > v(a)$. Observe also that v induces a short exact sequence

$$0 \longrightarrow \bar{D}^{\boldsymbol{\cdot}} \xrightarrow{\ f\ } D^{\boldsymbol{\cdot}}/(1+M) \xrightarrow{\ g\ } \Gamma \longrightarrow 0. \tag{1.1}$$

(Here g is the map induced by $v: D^{\boldsymbol{\cdot}} \to \Gamma$, and f is the composition of the isomorphism $\bar{D}^{\boldsymbol{\cdot}} \to U/(1+M)$ and the inclusion $U/(1+M) \to D^{\boldsymbol{\cdot}}/(1+M)$.) Thus, congruence mod $1+M$ is exactly what is detected by the value group together with the residue division ring.

Now suppose D has an involution $*$. That is, $*$ is an antiautomorphism of D of period 2 (or period 2 or 1 if D is commutative). The involution is said to be of the *first kind* if it restricts to the identity map on the center F; otherwise $*$ is of the *second kind*. We can now define the basic class of valuations considered here.

DEFINITION (of a c-valuation). Let D be a division ring with involution $*$. We say that a valuation $v: D \to \Gamma$ is a *c-valuation* (with respect to $*$) if for all $a, a_i, b \in D^{\cdot}$,

(C1) $v(a^*) = v(a)$;

(C2) $v(\sum_i a_i a_i^*) = \min_i \{v(a_i a_i^*)\}$;

(C3) $b(aa^*) \equiv (aa^*)\, b (\mathrm{mod}\ 1 + M)$.

c-valuations arose originally as valuations associated to c-orderings on involutorial division algebras. (But, it has recently been learned that not every valuation ring of a c-ordering is the ring of a c-valuation.) For background on c-valuations and c-orderings, see [C_1, C_2]. For instance, if v is the trivial valuation on D, then v is a c-valuation if and only if (i) D is $*$-formally real (i.e., -1 is not expressible as a sum of norms xx^*) and (ii) either (a) $D = F$ or (b) $[D:F] = 4$ and $*$ is the standard involution of the quaternion algebra D; i.e., the unique involution of D for which $x \in D$ is symmetric ($x = x^*$) if and only if $x \in F$.

Before considering c-valuations further, we recall some key facts about valued division algebras. The valuation $v: D \to \Gamma \cup \infty$ of course restricts to a valuation $v|_F$ on the center F of D. For the objects associated to $v|_F$ we write $V_F = V \cap F$, $M_F = M \cap F$, $\Gamma_F = v(F^{\cdot})$, and $\bar{F} = V_F/M_F$. We view $\bar{F} \subseteq \bar{D}$ by identifying $\bar{F}$ with its image in $\bar{D}$ under the canonical inclusion. In the sequel we refer to $[\bar{D}:\bar{F}]$ as the *residue degree* of v, and write $[\bar{D}:\bar{F}] = f_v$; we call Γ/Γ_F the *relative value group* of v, and write e_v for the *ramification index* $|\Gamma/\Gamma_F|$ of v. Since we are assuming $[D:F] < \infty$, it is well-known and easily proved (cf. [S, p. 21]) that

$$(\textit{inertial inequality}) \quad f_v, e_v < \infty \qquad \text{and} \qquad f_v \cdot e_v \leqslant [D:F]. \tag{1.2}$$

The "Ostrowski theorem" for valued division algebras, proved by Morandi in [M, Theorem 3], gives an improvement on the inertial inequality. We will be interested exclusively in the case of residue characteristic 0, where this theorem says

$$\text{if } \mathrm{char}(\bar{D}) = 0, \text{ then } f_v \cdot e_v = [D:F]. \tag{1.3}$$

While every valuation on D restricts to a valuation on F, a valuation on F need not extend to D. However, it is known (cf. [Er, Corollary 1] or [W]) that

$$\text{a valuation } F \text{ has at most one extension to } D. \tag{1.4}$$

In connection with condition (C3), we will need to know the center $Z(D^{\cdot}/(1+M))$ of $D^{\cdot}/(1+M)$. This was determined in [C_2, Theorem 2.1.1]:

$$\text{If } \operatorname{char}(\bar{D}) \nmid [D:F], \text{ then } Z(D^{\cdot}/(1+M)) = F^{\cdot}(1+M)/(1+M). \tag{1.5}$$

The proof of (1.5) uses the well-known result of Wedderburn [We] on the factorization over D of the minimal polynomial (over F) of any element of D.

We say that the valued division ring D is *Henselian* if $v|_F$ is a Henselian valuation of the field F. For background on Henselian valuations on fields and Henselization, see [R_2], [E], or [R_1]. Henselian division algebras have been studied recently in [JW_2, PY_1, PY_2]. We isolate the particular property of Henselian division algebras we will use in Section 2:

If D is Henselian and 2 is invertible in V, then for each $a \in M$ there is some $b \in F(a)$ such that $1+a=b^2$. (1.6)

This follows by applying Hensel's Lemma to the polynomial $f(x)=x^2-(1+a) \in F(a)[x]$, whose image in $\overline{V \cap F(a)}[x]$ is $(x-1)(x+1)$. The valuation $v|_{F(a)}$ is Henselian since it is an extension of the Henselian valuation $v|_F$ in a finite degree extension field of F.

By a recent result of P. Morandi, every valued finite-dimensional division algebra D has a Henselian closure $D_\Phi = D \otimes_F \Phi$, where the field Φ is the Henselization of F with respect to $v|_F$. It is shown in [M, Theorem 2] that

D_Φ is a division ring with a (unique) valuation extending v on D and also extending the Henselian valuation on Φ; moreover, D_Φ is an immediate extension of D (1.7)

The last assertion means that the canonical injections $\bar{D}^{\cdot} \hookrightarrow \overline{D_\Phi}^{\cdot}$ and $\Gamma_D \hookrightarrow \Gamma_{D_\Phi}$ are each surjective; equivalently (in view of (1.1)) the canonical map $D^{\cdot}/(1+M) \to D^{\cdot}_\Phi/(1+M_{D_\Phi})$ is an isomorphism.

Now assume D has a involution $*$. We analyze the axioms (C1) to (C3) for a valuation v on D to be a c-valuation. The first axiom merely states that $*$ maps V onto itself (i.e., V is *$*$-closed*). This axiom (C1) is essential for our framework. When (C1) holds the maximal ideal M of V is $*$-closed, as is $1+M$. Hence, congruence mod$(1+M)$ is compatible with $*$. Furthermore $\bar{D}$ carries an involution $(x+M)^* = x^* + M$ which we will refer to as the *residue involution*. Note that the residue of the center $\bar{F}$ is a $*$-closed subfield of $\bar{D}$, as is the center $Z(\bar{D})$ of $\bar{D}$. From (1.4) above we can determine precisely when (C1) holds:

LEMMA 1.8. *Let $S=\{s \in F \mid s^* = s\}$, the symmetric subfield of the center F of D. Then the valuation ring V of D is $*$-closed if and only if $v|_F$ is the* *only extension of $v|_S$ to a valuation of F. In particular, whenever $*$ is of the first kend (i.e., $S = F$) the axiom* (C1) *holds for v.*

Proof. The field F is Galois extension of its subfield S with Galois group $\{\mathrm{id}, *\}$. It is well-known (cf. [E, (14.1)]) that the Galois group acts transitively on the set of valuation rings of the larger field extending a given valuation ring of the base field. Hence, $V \cap F$ is the only extension of $V \cap S$ to F if and only if $(V \cap F)^* = V \cap F$. This certainly holds if $V^* = V$. Conversely, note that by (1.4), V is the unique extension of $V \cap F$ to a valuation ring of D, and V^* is the unique extension of $(V \cap F)^*$. Hence, whenever $(V \cap F)^* = V \cap F$, we have $V^* = V$. ∎

We turn to axiom (C2). Assuming (C1) holds it is easy to see that (C2) is equivalent to the ring-theoretic condition: for all $a_i \in V$, $1 + \sum a_i a_i^*$ is invertible in V. The latter is clearly equivalent to the residue condition $1 + \sum \overline{a_i a_i^*} \neq 0$ in $\bar{D}$, which means by definition, $\bar{D}$ is $*$-formally real. Summarizing,

> Assuming (C1) holds for v, v satisfies (C2) if and only if the residue division ring $\bar{D}$ is $*$-formally real (re the residue involution). (1.9)

Observe also,

> If v satisfies (C2), then $\bar{D}$ and D have characteristic zero. Hence, v is defectless; that is, we have the inertial equality $e_v f_v = [D:F]$. (1.10)

This follows immediately from (1.3) since (C2) implies that for any natural number n, $v(n1) = 0$; hence $\mathrm{char}(\bar{D}) = 0$.

Now consider the third axiom (C3). This axiom has a significant bearing on the relative value group Γ/Γ_F and also on the involution on the residue ring. For the next proposition let $S = \{s \in F \mid s = s^*\}$ and let Γ_S be the value group of $v|_S$.

PROPOSITION 1.11. *Suppose the valuation v on D satisfies axioms* (C1) *and* (C2) *with respect to the involution $*$. Then the following assertions are equivalent:*

(1) *v is a c-valuation.*

(2) *For each $a \in D^{\cdot}$, $aa^* \in F^{\cdot}(1 + M)$.*

(3) *The factor group Γ/Γ_S is an elementary abelian 2-group and for every $r \in \bar{D}$, if $r = r^*$ then $r \in \bar{F}^{\cdot}$.*

(4) *For each $r \in \bar{D}$, $rr^* \in \bar{F}$ and there are elements $\{d_i\}_{i \in I} \subseteq D^{\cdot}$ mapping onto a generating set of Γ_D, such that each $d_i d_i^* \in F^{\cdot}(1 + M)$.*

Proof. (1)$\Leftrightarrow$(2). Condition (C3) says that aa^* maps to the center of $D^{\cdot}/(1+M)$, for each $a \in D^{\cdot}$. Thus, (1)$\Leftrightarrow$(2) is immediate from (1.5). (2)$\Rightarrow$(4) is clear. (4)$\Rightarrow$(2). Let $G = \{a \in D^{\cdot} \mid aa^* \in F^{\cdot}(1+M)\}$. Easy computations show G is closed under multiplication and inverses; so G is a subgroup of $D^{\cdot}$. Recall that $U = \{u \in D^{\cdot} \mid v(u) = 0\}$. The first part of (4) shows $U \subseteq G$, and the second part shows G/U maps onto $\Gamma \cong D^{\cdot}/U$. Hence, $G = D^{\cdot}$, proving (2).

(2)$\Rightarrow$(3). For any $\gamma \in \Gamma$ pick $a \in D^{\cdot}$ with $v(a) = \gamma$. By (2), $aa^* = cu$ for some $c \in F$ and $u \in 1+M$. Then $cu = aa^* = (aa^*)^* = c^*u^*$. So $c + c^* = c + cuu^{*-1} = c(1+uu^{*-1})$. Since $uu^{*-1} \in 1+M$, $1+uu^* \equiv 2 \pmod{M}$, so $v(1+uu^{*-1}) = 0$ as char $(\bar{D}) \neq 2$. Thus,

$$2\gamma = v(a^2) = v(aa^*) = v(c) = v(c(1+uu^{*-1})) = v(c+c^*) \in \Gamma_S.$$

This shows Γ/Γ_S is a 2-torsion abelian group. Now take any $r \in \bar{D}$. From condition (2) we have $rr^* \in \bar{F}$. Likewise $\bar{F}$ contains $(1+r)(1+r^*) = 1 + (r+r^*) + rr^*$; so $r + r^* \in \bar{F}$. Therefore, if $r = r^*$ we have $r = (r+r^*)/2 \in \bar{F}$, proving (3).

(3)$\Rightarrow$(2). Take any $a \in D^{\cdot}$. Since Γ/Γ_S is an elementary abelian 2-group, $v(aa^*) = v(a^2) \in 2\Gamma \subseteq \Gamma_S$. That is, there is an $s \in S^{\cdot}$ and $u \in U$ satisfying $aa^* = us$. Then $us = aa^* = (aa^*)^* = u^*s$, so $u = u^*$. By the first part of (3), $\bar{u} \in \bar{F}$, so $u \in F^{\cdot}(1+M)$; hence $aa^* = us \in F^{\cdot}(1+M)$. ∎

For later use we record a convenient form of the conditions for a c-valuation:

COROLLARY 1.12. *A valuation v on D is a c-valuation re $*$ if and only if the following conditions all hold*:

(i) *$v|_S$ has only one extension to F, where $S = \{s \in F \mid s = s^*\}$.*

(ii) *$\bar{D}$ is $*$-formally real with respect to the residue involution.*

(iii) *For every $r \in \bar{D}$, if $r = r^*$ then $r \in \bar{F}$.*

(iv) *Γ/Γ_S is an elementary abelian 2-group.*

Proof. Combine Lemma 1.8, (1.9), and Proposition 1.11. ∎

The next corollary gives some quick consequences of Proposition 1.11 which are fully proved in [C_2].

COROLLARY 1.13. *Let v be a c-valuation on D. Then,*

(1) *Γ/Γ_F is an elementary abelian 2-group. Hence, the ramification index e_v is a power of 2.*

(2) *The residue degree f_v equals 1, 2, or 4. Further,*

(a) *If* $f_v = 2$ *then* $\bar{D}$ *is a field, and the residue involution is of the second kind.*

(b) *If* $f_v = 4$ *then* $\bar{D}$ *is a quaternion algebra with center* $\bar{F}$, *and the residue involution is the standard involution of the first kind.*

(3) $[D:F]$ *is a power of* 4.

All the assertions of 1.13.2 follow easily from the observation that every $r \in \bar{D}$ is a root of a polynomial of degree 2 (namely $(x-r)(x-r^*)$) over the symmetric subfield of $\bar{F}$. Note that 1.13.3 follows at once from (1) and (2) and (1.10).

COROLLARY 1.14. *Let* $D \subseteq E$ *be division rings each finite-dimensional over its center. Suppose E has an involution* $*$ *which restricts to an involution of D, and suppose E has a* $*$*-valuation v with respect to which E is an immediate extension of D. Then v is a c-valuation on E if and only if v restricts to a c-valuation of D.*

Proof. We write M_D (resp. M_E) for the maximal ideal of the valuation on D (resp. E). The assumption that v is immediate over $v|_D$ means that $\bar{D} = \bar{E}$, $\Gamma_D = \Gamma_E$, and hence the canonical map $D^{\cdot}/(1+M_D) \to E^{\cdot}/(1+M_E)$ is an isomorphism.

Now, $*$ on D induces an antiautomorphism (also called $*$) of the group $D^{\cdot}/(1+M_D)$. Condition (C3) is clearly equivalent to: for each $\delta \in D^{\cdot}(1+M_D)$, $\delta\delta^* \in Z(D^{\cdot}/(1+M_D))$. Since the isomorphism $D^{\cdot}/(1+M_D) \to E^{\cdot}/(1+M_E)$ is involution-preserving, (C3) holds for v on E if and only if it holds for v on D. Likewise, since $\bar{D} = \bar{E}$ with the same involution, (1.9) shows that (C2) holds for v on E if and only if it holds for $v|_D$. Property (C1) holds for v on E and for $v|_D$ by assumption. ▮

2. DECOMPOSABILITY OF HENSELIAN c-VALUED DIVISION RINGS

We can now prove that Henselian c-valued division rings decompose into a tensor product of quaternion algebras. Throughout this section D stands for a division ring finite-dimensional over its center F, and D has an involution $*$ which may be of the first or second kind. Assume also D has a valuation v. The associated terminology V, M, U, Γ, $\bar{D}$, etc., is as defined in Section 1. Recall that condition (H) in Theorem 2.1 below holds whenever $v|_F$ is Henselian (cf. (1.6)).

THEOREM 2.1 (Main Theorem). *Let D be any finite-dimensional c-valued division ring with* $D \neq F$, *and suppose that*

(H) *given any* $a \in M$ *there is a* $b \in F(a)$ *such that* $1 + a = b^2$.

Then D decomposes into a tensor product of $*$-closed quaternion algebras over F.

Proof. Suppose A is a $*$-closed subalgebra of D with the same center F. If A' is the centralizer of A in D, we have $D = A \cdot A' \cong A \otimes_F A'$. Observe that A' is $*$-closed, that $v|_{A'}$ is a $\mathcal{C}$-valuation of A', and that (H) holds for A'. Hence, by induction on $[D:F]$ it suffices to show that D contains a $*$-closed F-central quaternion subalgebra A.

If every symmetric element $s = s^*$ of D lies in F, then every element of D has degree at most 2 over F. (For, $a \in D$ is a root of $x^2 - (a + a^*)x + a^*a \in F[x]$.) Hence, $[D:F] \leqslant 4$, so that D is itself a quaternion algebra over F there is nothing more to show. Thus, we may assume there is an $s = s^* \in D$ with $s \notin F$. If the minimal polynomial of s over F is $x^n + c_{n-1}x^{n-1} + \cdots + c_0$, then each $c_i = c_i^* \in F$. So, by replacing s by $s + (1/n)c_{n-1}$, we may assume $c_{n-1} = 0$. It then follows that $s \notin F^{\cdot}(1+M)$. For, if $s = cu$ with $c \in F^{\cdot}$, $u \in 1 + M$, then every conjugate of s has the form cu' with $u' \in 1 + M$. Since $-c_{n-1}$ is the sum of n such conjugates by Wedderburn's factorization of the minimal polynomial of s [We], $v(c_{n-1}) = v(ns) = v(s) < \infty$, contradicting $c_{n-1} = 0$.

Because v is a $\mathcal{C}$-valuation we have from Proposition 1.11,

(i) $s^2 = ss^* = z(1 + a)$ for some $z \in F^{\cdot}$ and $a \in M$.

By replacing z by $(z + z^*)/2$ as in the proof of Proposition 1.11, we may assume that $z = z^*$. We rewrite (i) as

(ii) $a = z^{-1}s^2 - 1$.

Since both z and s are symmetric and z is central, $a = a^*$. By assumption (H) there is a $b \in F(a)$ with $b^2 = 1 + a$. Then, $b^* \in F(a)$ and $b^{*2} = b^2$. Hence, $b^* = \pm b$. But, because $b^2 \equiv 1 \pmod{M}$, we have $b \equiv \pm 1 \pmod{M}$. This shows $b^* \equiv b \pmod{M}$, yielding $b^* = b$. Note also that $b \in F(a) \subseteq F(s)$ by (ii); so $bs = sb$.

Put $s_1 = sb^{-1}$. Then $s_1 = s_1^*$ since s and b are commuting symmetric elements. Further, $s_1^2 = s^2 b^{-2} = z \in F$. In addition, we must have $s_1 \notin F$. For, $s_1 \in F$ would yield $s = s_1 b = s_1(\varepsilon + m) = s_1\varepsilon(1 + \varepsilon m) \in F^{\cdot}(1 + M)$, which we earlier ruled out. Thus, after replacing s by s_1, we may assume we have $s \in D$ with $s = s^*$, $s \notin F$, and $s^2 \in F$.

Write $[a, b]$ for the commutator $ab - ba$. If our symmetric s were to commute with all skew-symmetric elements of D, then $[s, [s, a]] = 0$ for all $a \in D$ (since this holds if $a = a^*$ or $a = -a^*$). Then, as 2 is a unit of D, which is prime, a theorem of Herstein [H, Lemma 1.1.9] says s would be central. So, since we know $s \notin F = Z(D)$, there must be a $k \in D^{\cdot}$ with

$k^* = -k$ and $sk - ks \neq 0$. Let $d = sk - ks \in D^{\cdot}$, then $d^* = k^*s^* - s^*k^* = d$. Furthermore, since s^2 is central,

$$sd + ds = s^2k - sks + sks - ks^2 = 0,$$

so $ds = -sd$.

By working with d as we did earlier with s, we can write $d^2 = z'(1 + a')$ with $z' = z'^* \in F^{\cdot}$ and $a' = a'^* \in M$. Then $1 + a' = b'^2$ with $b' \in F(a')$. Set $d_1 = db'^{-1}$. Just as before, we have $b' = b'^*$ and $db' = b'd$, so that $d_1 = d_1^*$ and $d_1^2 = z' \in F^{\cdot}$. Now, the centralizer of s contains d^2 (as $sd = -ds$), hence also a' (as $d^2 = z'(1 + a')$), hence also b' (as $b' \in F(a')$). Thus, for $d_1 = db'^{-1}$, we have $sd_1 = -d_1 s$.

All in all we have found a pair of anticommuting (hence noncentral) symmetric elements s and d_1 with s^2, $d^2 \in F^{\cdot}$. Put $A = F + F_s + Fd_1 + Fsd_1$. It is clear that A is a 4-dimensional $*$-closed F-central division ring in D. Hence, A is a quaternion algebra over F, and the theorem follows by induction. ∎

Remark. As the referee has pointed out, the proof of Theorem 2.1 yields a little more information in case the involution on D is of the second kind. If $S = \{s \in F \mid s = s^*\}$, then in fact D decomposes into a tensor product of quaternion algebras each of which is a scalar extension of a $*$-closed quaternion algebra over S. (For, in the argument above, $A = A_1 \otimes_S F$ where $A_1 = S + Ss + Sd_1 + Ssd_1$, and s^2, $d_1^2 \in S$ as $s^* = s$, $d_1^* = d_1$.) This holds whenever the valuation on F is Henselian, but is not true in general—see Remark (ii) after the proof of Theorem 4.1.

By combining Theorem 2.1 with earlier results we obtain the interesting

THEOREM 2.2. *Every finite-dimensional c-valued division ring D has an immediate extension D_Φ which decomposes into a tensor product of $*$-closed quaternionic division subrings.*

Proof. Let $S = \{s \in F \mid s = s^*\}$, the symmetric subfield of F, and let Ψ be the Henselization of S with respect to $v|_S$. Set $D_\Psi = D \otimes_S \Psi$. The S-linear involution $*$ of D and the identity map $id\colon \Psi \to \Psi$ (which is also an S-linear involution) combine to yield an involution $* \otimes id\colon D \otimes_S \Psi \to D \otimes_S \Psi$. Note that the center of D_Ψ is $F \otimes_S \Psi$, and the symmetric subfield of $* \otimes id$ on $F \otimes_S \Psi$ is clearly $S \otimes_S \Psi \cong \Psi$. Also, $* \otimes id$ restricts to $*$ on D (i.e., $D \otimes_S S$) in D_Ψ.

Because F is separable of finite degree over S, $F \otimes_S \Psi$ is a direct sum of fields, which are in one-to-one correspondence with the extensions of $v|_S$ to F, and each summand is a Henselization of the corresponding extension of $v|_S$ to F (cf. [E, (17.17)]). But since v is $*$-closed we have seen in Lemma 1.8 that $v|_F$ is the unique extension of $v|_S$ to F. Therefore, $F \otimes_S \Psi$

is a field, namely the Henselization Φ of $v|_F$. Hence, $D_\Psi = D \otimes_S \Psi \cong D \otimes_F (F \otimes_S \Psi) \cong D \otimes_F \Phi = D_\Phi$. By translating over to D_Ψ the information given by Morandi's theorem on D_Φ in (1.7) above, we see that D_Ψ is a division ring with valuation v' which is an immediate extension of v on D. This valuation restricts to the Henselization of $v|_F$ on $F \otimes_S \Psi$, and so to the Henselization of $v|_S$ on Ψ. Since $v'|_\Psi$ is Henselian, it has a unique extension to $F \otimes_S \Psi$. Therefore, Lemma 1.8 shows that v' is a $*$-valuation of D_Ψ. By Corollary 1.14, v' is a c-valuation of D_Ψ. By Theorem 2.1 (and (1.6)), D_Ψ decomposes into a tensor product of quaternion algebras, and so must the isomorphic algebra D_Φ decompose. ∎

Theorem 2.2 confirms the impression noted in the Introduction that every c-valued division ring "looks like" a tensor product of quaternion algebras in terms of its valuation theory. However, we will show in Section 4 that not every c-valued division algebra admits such a decomposition.

3. Constructions of c-Valued Division Algebras

In Section 4 we will give examples of c-valued division algebras which do not decompose into tensor products of quaternion algebras. We now build up a basic stock of c-valued division algebras, for use in the constructions on Section 4. We first consider totally ramified division algebras and their tensor products with other c-valued division algebras. Then we will treat quaternion algebras. We will give in Theorems 3.8 and 3.9 necessary and sufficient criteria for a valuation on the center F of a quaternion algebra Q to extend to a c-valuation of Q with respect to the standard involution. Our criteria are expressed in terms of easily checked residue conditions. By way of comparison, note that criteria (not involving valuations) for an ordering of F to extend to a c-ordering of Q were given in [C_2, Theorem 3.3.6].

Recall that a finite-dimensional valued division algebra D is said to be *totally ramified* (over its center F) just when $e_v = [D:F]$. Then, by the inertial inequality $f_v = 1$, i.e., $\bar{D} = \bar{F}$. The theory of totally ramified division algebras is developed in [TW].

THEOREM 3.1. *Let D be a valued division algebra finite-dimensional and totally ramified over its center F. Suppose $\bar{F}$ is formally real. Then,*

(1) *Γ/Γ_F is an elementary abelian 2-group, and $aba^{-1}b^{-1} \equiv \pm 1$ (mod M) for every $a, b \in D^{\cdot}$.*

(2) *If D has an involution $*$ of the first kind, then v is a c-valuation of D re $*$.*

Proof. (1) It is shown in [TW, Sect. 3] that for any tame and totally ramified division algebra D, there is a well-defined $\mathbb{Z}$-bilinear symplectic canonical pairing $B\colon \Gamma/\Gamma_F \times \Gamma/\Gamma_F \to \mu(\bar{F})$ given by $(v(a)+\Gamma_F, v(b)+\Gamma_F) \mapsto \overline{aba^{-1}b^{-1}}$ for all $a, b \in D^{\bullet}$. Here, $\mu(\bar{F})$ denotes the group of all roots of unity of $\bar{F}$. Since $\bar{F}$ is formally real, $\mu(\bar{F}) = \{\pm 1\}$. That is, for any $a, b \in D^{\bullet}$, $aba^{-1}b^{-1} \equiv \pm 1 \pmod{M}$. Hence, $a^2ba^{-2}b^{-1} \equiv 1 \pmod{M}$, showing a^2 maps to $Z(D^{\bullet}/(1+M))$; so, by (1.5), $a^2 \in F^{\bullet}(1+M)$. Then $2v(a) = v(a^2) \in \Gamma_F$. This shows Γ/Γ_F is an elementary abelian 2-group.

(2) Because $*$ is of the first kind, v is $*$-invariant by Lemma 1.8. Since $\bar{D} = \bar{F}$ the residue involution is trivial, so $\bar{D}$ is $*$-formally real by hypothesis. Thus, we have (C1) and (C2). For (C3), take any $a \in D^{\bullet}$. Since $v(aa^*) = v(a^2) \in \Gamma_F$, the canonical pairing shows $\overline{aa^*b(aa^*)^{-1}b^{-1}} = 1$ for all $b \in D^{\bullet}$; i.e., aa^* maps to $Z(D^{\bullet}/(1+M))$. Thus, v is a c-valuation. ∎

COROLLARY 3.2. *Let D be as in Theorem 3.1. Then the Henselization D_Φ of D has an involution of the first kind, with respect to which v is a c-valuation.*

Proof. Since D_Φ is an immediate extension of D, the hypotheses of Theorem 3.1 carry over from D to D_Φ. But since v on Φ is Henselian and D_Φ is totally ramified Draxl's decomposition theorem [D, Theorem 1] (or see [TW, Theorem 4.7(i)]) shows that D decomposes into a tensor product of symbol algebras each of degree dividing the exponent of Γ/Γ_F. By Theorem 3.1.1, Γ/Γ_F is an elementary abelian 2-group. Hence, the tensor factors of D_Φ are quaternion algebras, and D_Φ has order 2 (or 1) in the Brauer group $\mathrm{Br}(\Phi)$. By Albert's theorem D_Φ has an involution $*$ of the first kind. Theorem 3.1.2 shows v is a c-valuation of D_Φ re $*$. ∎

Several of the c-valued division algebras considered in this section and the next will be constructed as tensor products. To see that a valuation extends to a tensor product, we will frequently invoke the following result of Morandi [M, Theorem 1]:

PROPOSITION 3.3. *Let D_1 and D_2 be division rings, and F a field with $F \subseteq Z(D_i)$, $i = 1, 2$. Suppose each D_i has a valuation v_i and that v_1 and v_2 have the same restriction to F. Suppose further,*

(i) $\Gamma_{D_1} \cap \Gamma_{D_2} = \Gamma_F$;

(ii) $\overline{D_1} \otimes_{\bar{F}} \overline{D_2}$ *is a division ring*;

(iii) $[D_1 : F] < \infty$ *and D_1 is defectless over F re v_1 (i.e., $[D_1 : F] = [\overline{D_1} : \bar{F}]\,|\Gamma_{D_1} : \Gamma_F|$).*

Then $D_1 \otimes_F D_2$ is a division ring and there is a (unique) valuation on $D_1 \otimes_F D_2$ extending both v_1 on D_1 and v_2 on D_2. Furthermore, $\Gamma_{D_1 \otimes_F D_2} = \Gamma_{D_1} + \Gamma_{D_2}$ and $\overline{D_1 \otimes_F D_2} \cong \overline{D_1} \otimes_{\bar{F}} \overline{D_2}$.

THEOREM 3.4. *Let E be a finite-dimensional F-central division algebra with involution $*_E$ of the first kind and c-valuation v_E. Let T be a division algebra finite-dimensional over its center $Z(T)$ with involution $*_T$ of the first kind, and with a valuation v_T such that T is totally ramified over $Z(T)$. Suppose $F \subseteq Z(T)$, v_E and v_T restrict to the same valuation on F, $\bar{T} = \bar{F}$, and $\Gamma_T \cap \Gamma_E = \Gamma_F$. Let $D = E \otimes_F T$. Then Proposition 3.3 applies to D, and the valuation v_D on D extending v_E and v_T is a c-valuation with respect to the involution $*_E \otimes *_T$. Furthermore, D has the same residue degree over its center $Z(D) = Z(T)$ as E has over F.*

Proof. Proposition 3.3 applies to D since $\bar{E} \otimes_{\bar{F}} \bar{T} = \bar{E} \otimes_{\bar{F}} \bar{F} \cong \bar{E}$ which is a division ring, and E is defectless over F by (1.10). Let $* = *_E \otimes *_T$, which is clearly an involution of the first kind of D. Note that $Z(D) = Z(E) \otimes_F Z(T) \cong Z(T)$ as $Z(E) = F$. To see that v_D is a c-valuation of D re $*$ we verify conditions (i)–(iv) of Corollary 1.12. Condition (i) holds by Lemma 1.8, as $*$ is of the first kind. Conditions (ii) and (iii) carry over from E to D as $\bar{D} = \bar{E}$ and D and E have the same residue involution. Condition (iv) says that $\Gamma_D / \Gamma_{Z(T)}$ is an elementary abelian 2-group. But $\Gamma_D / \Gamma_{Z(T)} = (\Gamma_E + \Gamma_T) / \Gamma_{Z(T)}$, which is a homomorphic image of $(\Gamma_E / \Gamma_F) \oplus (\Gamma_T / \Gamma_{Z(T)})$. The first direct summand is elementary abelian by (1.12)(iv) for v_E, and the second direct summand is elementary abelian by Theorem 3.1 for v_T. ∎

We next give a device for building c-valued division algebras with involution of the second kind from c-valued algebras with involution of the first kind.

LEMMA 3.5. *Let E be a finite-dimensional F-central division algebra with involution of the first kind and c-valuation v_E. Suppose there is an $a \in F^{\bullet}$ with $v_E(a) \notin 2\Gamma_E$. Let $L = F(\sqrt{a})$, and let σ be the nonidentity F-automorphism of L. Let $D = E \otimes_F L$. Then D is a division ring and v_E extends to a valuation v_D of D with $\bar{D} \cong \bar{E}$ and $\Gamma_D = \Gamma_E + \langle \frac{1}{2} v_E(a) \rangle$. Moreover, v_D is a c-valuation of D with respect to the involution $*_E \otimes \sigma$ of the second kind.*

Proof. Since $v_E(a) \notin 2\Gamma_F$ the restriction to F of the valuation v_E has a unique extension to L which is totally ramified over F with $\Gamma_L = \Gamma_F + \langle \frac{1}{2} v_E(a) \rangle$ and $\bar{L} = \bar{F}$. Then, as $\bar{E} \otimes_{\bar{F}} \bar{L} \cong \bar{E}$ is a division algebra and $\Gamma_E \cap \Gamma_L = \Gamma_F$ (otherwise $\frac{1}{2} v_E(a) \in \Gamma_E$) Morandi's product theorem, Proposition 3.3, applies to show that D is a division ring with $\bar{D} \cong \bar{E} \otimes_{\bar{F}} \bar{L} \cong \bar{E}$ and $\Gamma_D = \Gamma_E + \Gamma_L = \Gamma_E + \langle \frac{1}{2} v_E(a) \rangle$. Of the conditions (i)–(iv) for a c-valuation given in Corollary 1.12, (ii) and (iii) hold for v_D as $\bar{D}=\bar{E}$ with the same residue involution. We have $Z(D)=L$ and the symmetric subfield of $Z(D)$ is F. (This is the S of Corollary 1.12.) Condition (i) holds since we have already noted that the valuation of F has a unique extension to L. Condition (iv) says that Γ_D/Γ_F is an elementary abelian 2-group. This holds as $\Gamma_D/\Gamma_F=(\Gamma_E+\Gamma_L)/\Gamma_F$, with Γ_E/Γ_F elementary abelian by (1.12) (iv) for v_E, and Γ_L/Γ_F clearly elementary abelian. Thus, v_D is a c-valuation. ∎

We now turn to c-valuations on quaternion algebras. Let Q be the quaternion algebra $(\frac{a,b}{F})$, with F a field ($\text{char}(\bar{F})\neq 2$), and $a, b\in F^{\cdot}$. Then Q has its standard F-base $\{1, i, j, ij\}$, where $i^2=a$, $j^2=b$, and $ij=-ji$. We work only with the *standard involution* $*$ on Q, which is the unique involution on Q for which the set of symmetric elements coincides with the center. Thus, $*$ is of the first kind, and $i^*=-i$, $j^*=-j$, and $(ij)^*=-(ij)$.

LEMMA 3.6. *The quaternion algebra* $Q=(\frac{a,b}{F})$ *is* $*$*-formally real (with respect to its standard involution* $*$*) if and only if there is an ordering of* F *at which both* a *and* b *are negative.*

Proof. If Q is $*$-formally real, let $T=\{\sum c_ic_i^* \mid c_i\in Q\}\subseteq F$. Then T is closed under addition and multiplication, $F^2\subseteq T$, and, as Q is $*$-formally real, $T\cap -T=\{0\}$. That is, T is a preordering of F. From the theory of ordered fields (cf. [P, Sect. 1]) it is known that T lies in a maximal preordering P of F, and P defines an ordering of F by $c\leqslant d$ if and only if $d-c\in P$. With respect to this ordering $a<0$ and $b<0$ as $-a=ii^*\in T\subseteq P$ and $-b=jj^*\in P$. Conversely, suppose there is an ordering $<$ of F with $a<0$ and $b<0$. Then, for any $c=r+si+tj+uij\in D^{\cdot}$ ($r, s, t, u\in F$), $cc^*\neq 0$ and $cc^*=r^2-s^2a-t^2b+u^2ab>0$. Hence, for $c_i\neq 0$, $\sum c_ic_i^*>0$, showing that Q is $*$-formally real. ∎

For quadratic field extensions we have the following analogue to the previous lemma, which is proved in just the same way.

LEMMA 3.7. *Let* $L=F(\sqrt{a})$ *where* F *is a field and* $a\in F-F^2$, $\text{char}(F)\neq 2$. *Let* $*$ *be the nonidentity* F*-automorphism of* L. *Then* L *is* $*$*-formally real if and only if there is an ordering of* F *with* $a<0$.

We can now describe when a valuation on a quaternion algebra is a c-valuation with respect to the standard involution.

THEOREM 3.8. *Let* F *be a field with valuation* v, *let* $a, b\in F^{\cdot}$ *with* $v(a)=v(b)=0$, *and let* $Q=(\frac{a,b}{F})$. *Then the following are equivalent:*

(i) *Q is a division ring and v extends to a c-valuation of Q with respect to the standard involution $*$.*

(ii) *There is an ordering of $\bar{F}$ with $\bar{a}<0$ and $\bar{b}<0$.*

Proof. We first verify that (assuming $v(a)=v(b)=0$ and $\operatorname{char}(\bar{F})\neq 2$) v extends to a valuation on Q if and only if $(\frac{\bar{a},\bar{b}}{\bar{F}})$ is a division ring, and when this occurs $\bar{Q}=(\frac{\bar{a},\bar{b}}{\bar{F}})$ and $\Gamma_Q=\Gamma_F$. For this, let $Q'=(\frac{\bar{a},\bar{b}}{\bar{F}})$ and let $A=$ "$(\frac{a,b}{V_F})$," the subring of Q generated over V_F by i and j. Then A is a free V_F-module of rank 4 with base $\{1, i, j, ij\}$ and $A/M_FA\cong Q'$. (So, A is an Azumaya algebra over V_F.) If v extends to a valuation of Q then the valuation ring V_Q equals $\{c\in Q\mid c$ is integral over $V_F\}$ by [W, Corollary]. So $V_Q\supseteq A$, as A is a finitely generated V_F-module. Furthermore, $M_FA\subseteq M_Q\cap A\subsetneq A$. Thus, there is an $\bar{F}$-homomorphism $A/M_FA\to V_Q/M_Q$, which must be injective since A/M_FA is simple; i.e., $Q'\subseteq\bar{Q}$. Then Q' must be a division ring, and the inertial inequality $e_v f_v\leqslant[Q:F]=4$ implies $\bar{Q}=Q'$ (as $[Q':\bar{F}]=4$) and $e_v=1$, i.e., $\Gamma_Q=\Gamma_F$.

Conversely, suppose Q' is a division ring. Then define a function $v\colon Q\to\Gamma_F\cup\infty$ by $v(r+si+tj+uij)=\min\{v(r), v(s), v(t), v(u)\}$. Clearly $v(c+d)\geqslant\min(v(c), v(d))$ for all $c, d\in Q$; and $v(rc)=v(r)+v(c)$ if $r\in F^{\cdot}$, $c\in Q$. Also, every $c\in Q-\{0\}$ is expressible as $c=rc'$ with $r\in F^{\cdot}$, $v(r)=v(c)$, and $v(c')=0$. Note further that $v(c)\geqslant 0$ if and only if $c\in A$, and $v(c)>0$ if and only if $c\in M_FA$. Now, $A/M_FA\cong Q'$ which is assumed to be a division ring. Hence, if $v(c)=0$ and $v(d)=0$, then $c, d\in A-M_FA$; so $cd\in A-M_FA$, i.e., $v(cd)=0$. Thus, for any $c, d\in Q-\{0\}$, if we write $c=rc'$ and $d=sd'$ with $r, s\in F^{\cdot}$ and $v(c')=v(d')=0$, then

$$v(cd)=v(rsc'd')=v(rs)+v(c'd')=v(rs)=v(r)+v(s)=v(c)+v(d).$$

This shows $cd\neq 0$; hence Q is a division ring. The equation also shows v is a valuation on Q.

We now prove (i) $\Leftrightarrow$ (ii) of Theorem 3.8. Suppose (i) holds. We have just shown that $\bar{Q}\cong Q'$. The residue involution is clearly the standard involution on $Q'=(\frac{\bar{a},\bar{b}}{\bar{F}})$. Since v is a c-valuation, Q' must be $*$-formally real. Lemma 3.6 shows there is an ordering of $\bar{F}$ with $\bar{a}<0$ and $\bar{b}<0$, proving (ii). Conversely, assume (ii) holds. By Lemma 3.6, Q' is $*$-formally real, hence a division ring. Therefore, v extends to a valuation of the division ring Q, as we just proved, and $\bar{Q}\cong Q'$. Property (C1) holds because the standard involution on Q is of the first kind; (C2) holds as $\bar{Q}$ is $*$-formally real, by (1.9); and (C3) holds as $cc^*\in F$ for each $c\in Q$. So v is a c-valuation of Q, proving (i). ∎

THEOREM 3.9. *Let F be a field with valuation v; let $a, b\in F^{\cdot}$ with $v(a)=0$ and $v(b)\notin 2\Gamma_F$; and let $Q=(\frac{a,b}{F})$. Then the following are equivalent:*

(i) *Q is a division ring and v extends to a c-valuation of Q with respect to the standard involution.*

(ii) *There is an ordering of $\bar{F}$ with $\bar{a} < 0$.*

Proof. We verify first that (assuming $\operatorname{char}(\bar{F}) \neq 2$) Q is a division ring and v extends to a valuation of Q if and only if $\bar{a} \notin \bar{F}^2$; further, when this occurs, $\Gamma_Q = \Gamma_F + \langle \frac{1}{2} v(b) \rangle$ and $\bar{Q} \cong \bar{F}(\sqrt{\bar{a}})$. For, if $\bar{a} \in \bar{F}^2$, then either $F(\sqrt{a})$ (i.e., $F[x]/(x^2 - a)$) is not a field or it is a field but v has two different extensions to it. Then, as $F(\sqrt{a})$ is a subring of Q, v cannot extend to Q, by [W, Theorem]. On the other hand, if $\bar{a} \notin \bar{F}^2$, then [JW_2, Example 4.3] (or an easy direct calculation) shows Q is a division ring and v extends to a valuation of Q, with $\bar{Q}$ and Γ_Q as claimed. The proof that (i)$\Leftrightarrow$(ii) now follows by the analogous argument to the preceding proof, with Lemma 3.7 replacing 3.6. ∎

Let $Q = \left(\frac{a,b}{F}\right)$ be any quaternion algebra over a valued field F. If $v(a)$ and $v(b)$ map to $\mathbb{Z}_2$-dependent elements of $\Gamma_F/2\Gamma_F$ then we can modify the choice of a and b in describing Q (multiplying by a square or replacing a or b by ab) to obtain $v(a) = 0$ and either $v(b) = 0$ or $v(b) \notin 2\Gamma_F$. These are the situations described in Theorems 3.8 and 3.9. The only remaining possibility is that $v(a)$ and $v(b)$ are $\mathbb{Z}_2$-independent in $\Gamma_F/2\Gamma_F$. In this case, Q is a division algebra and v extends to a valuation of Q totally ramified over F (cf. [TW, Proposition 3.5]). Then, by Theorem 3.1 v is a c-valuation for the standard valuation on Q if and only if $\bar{F}$ is formally real.

We have determined when a valuation v on a quaternion algebra Q is a c-valuation with respect to the standard involution $*$. It is not difficult to work out what happens with respect to the other involutions on Q of the first kind. For, every other such involution $*'$ on Q is obtained as $a^{*'} = ca^*c^{-1}$ for some c in the F span of $\{i, j, ij\}$. If $[\bar{Q}:\bar{F}] = 4$, v is never a c-valuation with respect to any such $*'$. If $[\bar{Q}:\bar{F}] = 2$, then v is c-valuation re $*'$ if and only if $*'$ induces the nonidentity $\bar{F}$-automorphism of $\bar{Q}$ and v is a c-valuation re the standard involution $*$. If $\bar{Q} = \bar{F}$, then we know from Theorem 3.1 that v is a c-valuation with respect to any $*'$ if and only if $\bar{F}$ is formally real. Note that in all cases if v is a c-valuation with respect to some first kind involution of Q then it is also a c-valuation with respect to the standard involution.

4. Indecomposable c-Valued Division Algebras

In this section we will exhibit examples first of c-valued division algebras which do not decompose into tensor products of quaternion algebras, and second of a c-valued division algebra which though it is a product of quaternion algebras, has no $*$-closed quaternion subalgebras. These examples show that the Henselian assumption made in our main theorem is genuinely needed.

An overview of the construction for the first examples is in order. Let $\mathrm{Br}_2(F)$ denote the 2-torsion subgroup of the Brauer group $\mathrm{Br}(F)$ of a field F. For our examples the ingredients are first, a division algebra in $\mathrm{Br}_2(F)$ which is not a tensor product of quaternion algebras. This is obtained by adapting the original such example given by Amitsur, Rowen, and Tignol in [ART]. Second, we need a division algebra of the same size, also in $\mathrm{Br}_2(F)$, which has a c-valuation by virtue of being totally ramified over F with formally real residue field (cf. Proposition 3.1). We smash these two algebras together so that the resulting division algebra has the specified properties of both the original algebras. This is done by the method used in [JW_1] to construct noncrossed product algebras. Thus, we will be working with two valuations, one to establish the indecomposability, and the other which is a c-valuation.

The source of most of the valuations we will use is the valuation on an iterated Laurent power series field, which we recall briefly. Let L be a field, and let $t_1, ..., t_n$ be independent indeterminates over L. Let $L((t_1))\cdots((t_n))$ denote the iterated Laurent power series field over L in $t_1, ..., t_n$. To get a valuation in this field we use the ordered group $\mathbb{Z}^n = \mathbb{Z} \times \cdots \times \mathbb{Z}$ (n times) with the right-to-left lexicographic ordering. (That is, $(i_1, ..., i_n) > (j_1, ..., j_n)$ just when there is an m, $1 \leqslant m \leqslant n$, such that $i_m > j_m$ and, for all $k > m$, $i_k = j_k$.) The *standard valuation* on $L((t_1))\cdots((t_n))$ is $v\colon L((t_1))\cdots((t_n))^{\cdot} \to \mathbb{Z}^n$ (ordered as above) defined by

$$v\left(\sum_{i_1}\cdots\sum_{i_n} c_{i_1\cdots i_n} t_1^{i_1}\cdots t_n^{i_n}\right) = \inf\{(i_1, ..., i_n) \mid c_{i_1\cdots i_n} \neq 0\}.$$

(So in particular, $v(t_1^{i_1}\cdots t_n^{i_n}) = (i_1, ..., i_n)$.) As is well known and easy to check, v is indeed a well-defined valuation with value group $\mathbb{Z}^n$ and residue field L. (This valuation is also Henselian, a fact we will not need.) Recall that the valuation ring V of v has Krull dimension n, and every valuation ring in $L((x_1))\cdots((x_n))$ containg V is the ring of the standard valuation on $L_m((x_{m+1}))\cdots((x_n))$ where $L_m = L((x_1))\cdots((x_m))$, for some m, $1 \leqslant m < n$. Note that when v is restricted to the rational function field $L(t_1, ..., t_n)$ the value group is still $\mathbb{Z}^n$ and the residue field still L; in addition, the maximal proper valuation ring of $L(t_1, ..., t_n)$ containing $V \cap L(t_1, ..., t_n)$ is the localization $k[t_1, ..., t_n]_{(t_n)}$ of the polynomial ring $k[t_1, ..., t_n]$ at the prime ideal (t_n).

Before launching into the constructions we recall some terminology for central simple algebras. If A is a central simple algebra over a field L (which is always understood to be finite dimensional over L) then by Wedderburn's theorem A is the $n \times n$ matrix ring $A \cong M_n(D)$ where D is a central simple division algebra over L. We call D (which is unique up to L-isomorphism) the *underlying division algebra* of A. The *degree* of A is $\sqrt{[A:F]}$ and the *Schur index* of A is the degree of D. We write $[A]$ for the image of A in the Brauer group $\mathrm{Br}(L)$. If B is another central simple L-algebra, we write $B \sim A$ if $[B]=[A]$. If $K \supseteq L$ is another field, $\mathrm{Br}(K/L)$ denotes the kernel of the scalar extension homomorphism $\mathrm{Br}(L) \to \mathrm{Br}(K)$, and $\mathrm{Br}_2(K/L)$ is the 2-torsion subgroup of $\mathrm{Br}(K/L)$. When K has the form $L(\sqrt{a_1}, ..., \sqrt{a_n})$ with $a_1, ..., a_n \in L^{\cdot}$, there is an important subgroup denoted $\mathrm{Dec}(K/L)$ which was introduced by Tignol (cf. $[\mathrm{T}_1]$). $\mathrm{Dec}(K/L)$ consists of the classes in $\mathrm{Br}(L)$ which decompose "according to K," i.e., the classes of algebras

$$\left(\frac{a_1, s_1}{L}\right) \otimes_L \cdots \otimes_L \left(\frac{a_n, s_n}{L}\right) \qquad \text{for some } s_1, ..., s_n \in L^{\cdot}.$$

(Tignol has shown $[\mathrm{T}_1$, Lemma 1.3$]$ that $\mathrm{Dec}(K/L)$ depends only on K and L, and not on the choice of the a_i.) The process used by Amitsur, Rowen, and Tignol in [ART] to construct a division algebra with involution which is not a tensor product of quaternion algebras was, essentially, first to find a triquadratic extension $K = L(\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3})$ with a division algebra D with $[D] \in \mathrm{Br}_2(K/L) - \mathrm{Dec}(K/L)$, and then to build a generic abelian crossed product out of D and the extension K/L. Because D does not decompose according to K the generic algebra does not decompose into quaternion algebras at all. The passage from D to the generic algebra can be readily interpreted in terms of valuation theory, as described in $[\mathrm{JW}_2$, Remark 5.16$]$. We will carry out an analogous process here.

We now give our counterexamples.

THEOREM 4.1. *There is a c-valued division ring D with involution of the first kind and finite-dimensional over its center which does not decompose into a tensor product of quaternion algebras.*

Proof. Let $k = \mathbb{Q}(t)$, where $\mathbb{Q}$ is the field of rational numbers and t is transcendental over $\mathbb{Q}$. Let $M = k(\sqrt{t}, \sqrt{t^2+1}, \sqrt{-1})$, so $[M:k]=8$. It is known (cf. [ART, Theorem 5.1] or, more explicitly, [ELTW, Theorem 5.1, Remark 5.8(i)]) that $\mathrm{Dec}(M/k) \subsetneq \mathrm{Br}_2(M/k)$. That is, there is a division algebra A over k with $[A] \in \mathrm{Br}_2(k)$, A is split by M, but A does not decompose according to M.

Let $k_1 = k(x_1, x_2, x_3)$, where the x_i are algebraically independent over k. Let B be the underlying division algebra of

$$(A \otimes_k k_1) \otimes_{k_1} \left(\frac{t, x_1}{k_1}\right) \otimes_{k_1} \left(\frac{t^2+1, x_2}{k_1}\right) \otimes_{k_1} \left(\frac{-1, x_3}{k_1}\right).$$

(This B is in fact the division algebra with involution proved in [ART] not to decompose into quaternion algebras.) We have $[B] \in \operatorname{Br}_2(k_1)$ and $[B:k_1] \leqslant 64$ as B is clearly split by $k_1 \cdot M$. But for our construction we need a formally real splitting field of B. For this, let $M_0 = k(\sqrt{t}, \sqrt{t^2+1})$. Since $[M:M_0] = 2$ it is well known that $\operatorname{Br}(M/M_0) = \operatorname{Dec}(M/M_0)$. Thus, as $M = M_0(\sqrt{-1})$ and M splits A, $A \otimes_k M_0 \sim (\frac{-1, \beta}{M_0})$ for some $\beta \in M_0^\cdot$. Then $B \otimes_{k_1} (k_1 \cdot M_0) \sim (\frac{-1, \beta x_3}{k_1 \cdot M_0})$. Let $k_2 = k_1(\sqrt{t}, \sqrt{t^2+1})(\sqrt{\beta x_3}) = k_1 \cdot M_0(\sqrt{\beta x_3})$. Then $[k_2 : k_1] = 8$ and k_2 clearly splits B. We claim that k_2 is formally real. To see this, take an ordering of $k = \mathbb{Q}(t)$ with $t > 0$. Then as $t^2 + 1 > 0$ this ordering extends to an ordering of $k(\sqrt{t}, \sqrt{t^2+1})$. This ordering can be extended to the purely transcendental extension $k(\sqrt{t}, \sqrt{t^2+1})(x_1, x_2, x_3)$ with any desired choice of sign for the indeterminates x_1, x_2, x_3. Choose the order extension so that x_3 has the same sign as β. Then as $\beta x_3 > 0$ this ordering extends to $k_2 = k(\sqrt{t}, \sqrt{t^2+1}, x_1, x_2, x_3)(\sqrt{\beta x_3})$, establishing the claim.

Now, let $K = k_1(y_1, ..., y_6)$, where the y_i are algebraically independent over k_1. Let

$$B_1 = B \otimes_{k_1} K \qquad \text{and} \qquad B_2 = \left(\frac{y_1, y_2}{K}\right) \otimes_K \left(\frac{y_3, y_4}{K}\right) \otimes_K \left(\frac{y_5, y_6}{K}\right).$$

Let v_1 be the valuation on K obtained by restriction from the standard valuation on $k(y_1, ..., y_6)((x_1))((x_2))((x_3))$. So, with respect to v_1, K has residue field $\bar{K}_{v_1} = k(y_1, ..., y_6)$ and value group $\Gamma_{K, v_1} = \mathbb{Z}^3$, with $v_1(x_1^{i_1} x_2^{i_2} x_3^{i_3}) = (i_1, i_2, i_3)$.

Let v_2 be the valuation on K obtained by restriction from the standard valuation on $k(x_1, x_2, x_3)((y_1)) \cdots ((y_6))$. So, with respect to v_2, K has residue field $\bar{K}_{v_2} = k(x_1, x_2, x_3) = k_1$ and value group $\Gamma_{K, v_2} = \mathbb{Z}^6$ with $v_2(y_1^{i_1} \cdots y_6^{i_6}) = (i_1, ..., i_6)$. Observe that v_1 and v_2 are independent on K (i.e., there is no proper valuation ring of K containing both the ring of v_1 and the ring of v_2). For, the maximal proper valuation ring of K containing the valuation ring of v_1 is the localization $k(y_1, ..., y_6)[x_1, x_2, x_3]_{(x_3)}$. But in every valuation ring containing that of v_2, x_3 is a unit.

We enlarge K to obtain the right residue fields. Let N be an extension field of K with $[N:K] = 8$, such that each v_i has a (unique) unramified extension (also called v_i) to N, with residue fields $\bar{N}_{v_1} = \bar{K}_{v_1}(\sqrt{y_1}, \sqrt{y_3}, \sqrt{y_5})$ and $\bar{N}_{v_2} = \bar{K}_{v_2}(\sqrt{t}, \sqrt{t^2+1})(\sqrt{\beta x_3})$. Such an N is obtainable as $N = K(\sqrt{\alpha_1}, \sqrt{\alpha_2})(\sqrt{\alpha_3})$, where $\alpha_1, \alpha_2 \in K^\cdot$ are chosen so that $v_1(\alpha_1 - y_1) > 0$, $v_2(\alpha_1 - t) > 0$, $v_1(\alpha_2 - y_3) > 0$, and $v_2(\alpha_2 - (t^2+1)) > 0$. The existence of such α_1 and α_2 is assured by the Approximation Theorem [E, (11.16)], since v_1 and v_2 are independent on K. Let $N_0 = K(\sqrt{\alpha_1}, \sqrt{\alpha_2})$. Then in any extension of v_2 to N_0, $v_2(\sqrt{\alpha_1}) = v_2(\alpha_1) = 0$ and $\sqrt{\alpha_1}$ maps to a square root of t in $\overline{N_0}_{v_2}$. Likewise, $\sqrt{\overline{\alpha_2}}=\sqrt{t^2+1}$ in $\overline{N_0}_{v_2}$. Thus, as $[\bar{K}_{v_2}(\sqrt{t},\sqrt{t^2+1}):\bar{K}_{v_2}]=[N_0:K]=4$, the fundamental inequality $\sum e_i f_i \leqslant [N_0:K]=4$ shows that v_2 has a unique and unramified extension from K to N_0, with $\overline{N_0}_{v_2}=\bar{K}_{v_2}(\sqrt{t},\sqrt{t^2+1})$. Likewise v_1 has a unique and unramified extension to N_0 with $\overline{N_0}_{v_1}=\bar{K}_{v_1}(\sqrt{y_1},\sqrt{y_3})$. The valuations v_1 and v_2 are independent on N_0, as N_0 is algebraic over K. Choose any $\gamma\in N_0$ with $v_2(\gamma_0)=0$ and $\bar{\gamma}=\beta x_3$ in $\overline{N_0}_{v_2}$. Then, by the Approximation Theorem again there is an $\alpha_3\in N_0$ with $v_1(\alpha_3-y_5)>0$ and $v_2(\alpha_3-\gamma)>0$. Let $N=N_0(\sqrt{\alpha_3})=K(\sqrt{\alpha_1},\sqrt{\alpha_2})(\sqrt{\alpha_3})$. By repeating the reasoning just used, we see that each v_i has a unique and unramified extension from N_0 to N with $\bar{N}_{v_1}=\overline{N_0}_{v_1}(\sqrt{y_5})=\bar{K}_{v_1}(\sqrt{y_1},\sqrt{y_3},\sqrt{y_5})$ and $\bar{N}_{v_2}=\overline{N_0}_{v_2}(\sqrt{\beta x_3})=\bar{K}_{v_2}(\sqrt{t},\sqrt{t^2+1})(\sqrt{\beta x_3})$, as desired.

In some fixed algebraic closure of N, let L_i be a Henselization of N with respect to v_i, $i=1,2$, and let $F=L_1\cap L_2$. Let D be the underlying division algebra of $(B_1\otimes_K B_2)\otimes_K F$. We'll show that D is the desired example! The method is to obtain information about $D\otimes_F L_1$ and $D\otimes_F L_2$, and use this to work back to D by invoking the local-global principles in [JW$_1$].

Observe that since v_1 on L_1 is Henselian and y_1, y_3, y_5 map to squares in $\overline{L_1}=\bar{N}_{v_1}$, we have $\sqrt{y_1},\sqrt{y_3},\sqrt{y_5}\in L_1$. Hence L_1 splits B_2; so $D\otimes_F L_1\sim B_1\otimes_K L_1$. Note that $B_1\otimes_K L_1\sim(A\otimes_k L_1)\otimes_{L_1} C$, where $C=\left(\frac{t,\,x_1}{L_1}\right)\otimes_{L_1}\left(\frac{t^2+1,\,x_2}{L_1}\right)\otimes_{L_1}\left(\frac{-1,\,x_3}{L_1}\right)$. Now, k injects into $\overline{L_1}=k(\sqrt{y_1},y_2,\sqrt{y_3},y_4,\sqrt{y_5},y_6)$, which is purely transcendental over k. Hence, $A\otimes_k\overline{L_1}$ is a division ring. It follows (e.g., by Proposition 3.3) that v_1 extends from L_1 to $A\otimes_k L_1$ with $A\otimes_k L_1$ unramified over L_1 with residue $\overline{A\otimes_k L_1}=A\otimes_k\overline{L_1}$. Note also that because $[A]\notin \operatorname{Dec}(M/k)$ and $\overline{L_1}$ is purely transcendental over k, $[A\otimes_k\overline{L_1}]\notin\operatorname{Dec}(\overline{L_1}\cdot M/\overline{L_1})$. This follows from Tignol's theorem [T$_1$, Corollary 1.5] that

$$\operatorname{Br}_2(M/k)/\operatorname{Dec}(M/k)\cong\operatorname{Br}_2(\overline{L_1}\cdot M/\overline{L_1})/\operatorname{Dec}(\overline{L_1}\cdot M/\overline{L_1}),$$

as $\overline{L_1}$ is purely transcendental over k.

Now, by [JW$_2$, Example 4.3] or easy direct calculation, C is a division ring with valuation extending v_1 on L_1 with $\bar{C}=\overline{L_1}(\sqrt{t},\sqrt{t^2+1},\sqrt{-1})$ and $\Gamma_C=(\frac{1}{2}\mathbb{Z})^3$. Further, C has a maximal subfield $L_1(\sqrt{t},\sqrt{t^2+1},\sqrt{-1})$ unramified over L_1 and another maximal subfield $L_1(\sqrt{x_1},\sqrt{x_2},\sqrt{x_3})$ which is a totally ramified Kummer extension of L_1. Thus, C is "nicely semiramified" in the terminology of [JW$_2$, Sect. 4]. Because $A\otimes_k L_1$ is unramified over L_1 and $\overline{A\otimes_k L_1}$ is split by $\bar{C}$, [JW$_2$, Theorem 5.15] says the underlying division algebra of $(A\otimes_k L_1)\otimes_{L_1} C$ (to which v_1 extends as v_1 is Henselian on L_1) has the same residue division algebra and value group, hence degree as C. Then, since $B_1\otimes_K L_1\sim(A\otimes_k L_1)\otimes_{L_1} C$ and $[B_1\otimes_K L_1:L_1]=[B:k_1]\leqslant 64=[C:L_1]$, $B_1\otimes_K L_1$ must be a division ring of degree 8, and is the underlying division algebra of $(A \otimes_k L_1) \otimes_{L_1} C$. Furthermore, because $\overline{A \otimes_k L_1} \notin \mathrm{Dec}(\overline{L_1} \cdot M/\overline{L_1})$ as we saw above, [JW$_2$, Proposition 4.8, Theorem 5.15(c)] shows $B_1 \otimes_K L_1$ is not isomorphic to a tensor product of quaternion algebras.

We turn now to L_2. The valuation v_2 on L_2 is Henselian with $\overline{L_2} = \bar{N}_{v_2} = \bar{K}_{v_2}(\sqrt{t}, \sqrt{t^2+1})(\sqrt{\beta x_3}) = k_2$ and $\Gamma_{L_2} = \Gamma_{N,v_2} = \Gamma_{K,v_2} = \mathbb{Z}^6$. Because v_2 is Henselian on L_2 and t, t^2+1, βx_3 map to squares of $\overline{L_2}$, they are squares in L_2. Therefore, L_2 splits B_1. So, $D \otimes_F L_2 \sim B_2 \otimes_K L_2$. The $\mathbb{Z}$-independence of the values of the y_i in Γ_{L_2} assures by [TW, Proposition 3.5] that $B_2 \otimes_K L_2$ is a division algebra with valuation totally ramified over L_2 re v_2, of degree 8 and exponent 2. In fact, $\Gamma_{B_2 \otimes_K L_2} = (\frac{1}{2}\mathbb{Z})^6$.

Because each (L_i, v_i) is an immediate extension of (N, v_i), (L_i, v_i) is an immediate extension of (F, v_i). Furthermore, v_1 and v_2 are independent on F, since they are independent on N, over which F is algebraic. Therefore, by [JW$_1$, Theorem 4.3], $G_2(F) \cong G_2(L_1) *_2 G_2(L_2)$, where $G_2(F)$ denotes the Galois group $\mathscr{G}(F(2)/F)$, where $F(2)$ is the maximal Galois 2-extension of F, and $*_2$ denotes the free product in the category of pro-2 groups. Therefore, the local-global principles given in [JW$_1$, Theorem 4.11] hold from L_1 and L_2 to F. We apply them to D. We have seen that for $i = 1, 2$, $D \otimes_F L_i \sim B_i \otimes_K L_i$ which has Schur index 8; further each $B_i \otimes_K L_i$ is evidently split by a Galois extension of degree 8 (so $B_i \otimes_K L_i$ has 2-index 8 in the terminology of [JW$_1$]). Therefore, by [JW$_1$, Theorem 4.11(ii), (iii)], D has Schur index (and degree) 8. Hence, $D \otimes_F L_i \cong B_i \otimes_K L_i$, $i = 1, 2$. Thus, since we have shown $B_1 \otimes_K L_1$ is not a tensor product of quaternion algebras, D cannot be isomorphic to a tensor product of quaternion algebras. On the other hand, the valution v_2 on $D \otimes_F L_2$ restricts to a valuation on D (restricting to v_2 on F). For the residues with respect to v_2 we have

$$\bar{N}_{v_2} \subseteq \bar{F}_{v_2} \subseteq \bar{D}_{v_2} \subseteq \overline{D \otimes_F L_2} = \overline{L_2} = \bar{N}_{v_2};$$

so $\bar{D}_{v_2} = \bar{F}_{v_2} = \bar{N}_{v_2}$, which we have arranged to be formally real. Morandi's Ostrowski theorem (1.3) above shows that D must be totally ramified over F re v_2. Since $[D] \in \mathrm{Br}_2(F)$ (as each $[B_i] \in \mathrm{Br}_2(K)$) Albert's theorem says that D has an involution $*$ of the first kind. Then, by Theorem 3.1, v_2 is a c-valuation with respect to $*$. Thus, D has the required properties. ∎

Remarks. (i) The method just used can also be applied to construct c-valued division rings of the second kind with the exponent of D in $\mathrm{Br}(F)$ any desired power of 2. (Of course, when the exponent exceeds 2, D is not a tensor product of quaternion algebras.)

(ii) The referee has pointed out the following: Take the D of Theorem 4.1, and any $d \in D^{\cdot} - F^{\cdot}$ with $d^2 \in F^{\cdot}$, choose an involution $*$ of the first kind on D with $d^* = -d$, and let C be the centralizer of d in D. Then the c-valuation on D restricts to a c-valuation on C with respect to the restriction of $*$, which is of the second kind on C. But C cannot have the form $A \otimes_F F(d)$ for any F-subalgebra A of D. For, if there were such an A, then D would be decomposable. (To see the existence of the prescribed d and $*$, first take, for example, $c = x_1^{-1} + y_1^{-1} \in F^{\cdot}$. Then, $D \otimes_F F(\sqrt{c})$ has Schur index 4 by [JW_1, Theorem 4.11(ii)], since $D \otimes_{L_1} L_1(\sqrt{c})$ has index 4 (as $c \equiv x_1^{-1} \pmod{L_1^{\cdot 2}}$) and likewise $D \otimes_{L_2} L_2(\sqrt{c})$ has index 4. So, there is a $d \in D^{\cdot}$ with $d^2 = c$. Now take any involution $*$ of D of the first kind. If $d^* \neq d$, let $k = d - d^*$, a skew element of $D^{\cdot}$. Then, (as $d^2 = d^{*2}$) $kd^* = -dk$, so that d is skew with respect to the involution $*'$ given by $a \mapsto k(a^*)k^{-1}$, as desired. On the other hand, if $d^* = d$, then since d is not central and $[D:F] > 4$ there is a skew k_1 with $k_1 d \neq dk_1$ (as in the proof of Theorem 2.1). Let $k = k_1 d - dk_1$, which is skew. Since $kd^* = -dk$, we can form $*'$ from $*$ and k as before, so that d is skew re $*'$.)

In the example just constructed D had residue degree 1 over F and the involution was of the first kind. We now show how to perform "minor surgery" on this example to obtain indecomposable c-valued division algebras of residue degrees 2 or 4, with involution of either kind. For this we begin with any division algebra D finite-dimensional over its center F with involution $*$ of the first kind and with a c-valuation v re $*$, such that D is not isomorphic to a tensor product of quaternion algebras. Let $K = F(x, y)$, where x and y are algebraically independent over F, and let $Q = \left(\frac{x, y}{K}\right)$, the quaternion algebra with its standard involution $*_Q$. Let $E = (D \otimes_F K) \otimes_K Q$, which is equipped with the involution $*_E = (* \otimes id) \otimes *_Q$ of the first kind. Also, let $L = F(x)$, and let $*_L$ be the nonidentity $F(x^2)$-automorphism of L. Let $A = D \otimes_F L$, which is given the involution $* \otimes *_L$ of the second kind.

PROPOSITION 4.2. *With D, F, K, E, L, A as above we have:*

(1) *E is a division algebra which is not isomorphic to a tensor product of quaternion algebras. If D is totally ramified over F, then E has a c-valuation re $*_E$ with residue degree* 2 *and another c-valuation with residue degree* 4.

(2) *A is a division algebra which is not isomorphic to a tensor product of quaternion algebras. Further, A has a c-valuation re the involution $* \otimes *_L$ of the second kind, with the same residue degree as that of D over F.*

Proof. (1) There are many ways of extending v on F to a valuation of K (cf. [B, Sect. 10, No. 1]). Let v_1 be the extension with $v_1(x) = v_1(y) = 0$,

$\bar{x}$ and $\bar{y}$ algebraically independent over $\bar{F}$, $\bar{K}_{v_1} = \bar{F}(\bar{x}, \bar{y})$, and $\Gamma_{K,v_1} = \Gamma_F$. ($v_1$ is defined on $F[x, y]$ by $v_1(\sum_i \sum_j c_{ij} x^i y^j) = \inf\{v(c_{ij})\}$.) Then, by Proposition 3.3, $D \otimes_F K$ is a division ring and v_1 extends to $D \otimes_F K$ with $\Gamma_{D \otimes_F K} = \Gamma_D + \Gamma_K = \Gamma_D$ and $\overline{D \otimes_F K} = \bar{D} \otimes_{\bar{F}} \bar{F}(\bar{x}, \bar{y}) = \bar{F}(\bar{x}, \bar{y}) = \bar{K}$. So $D \otimes_F K$ is totally ramified over K.

Because D is a c-valuation, $\bar{F}$ is formally real. Take any ordering on $\bar{F}$, and extend it to an ordering on $\bar{K}$ with $\bar{x} < 0$ and $\bar{y} < 0$. This ordering on $\bar{K}$ shows by Theorem 3.8 that v_1 is a c-valuation on Q with respect to $*_Q$, with residue degree 4. In addition as $\bar{K}$ is formally real and $D \otimes_F K$ is totally ramified over K, by Theorem 3.1, v_1 on $D \otimes_F K$ is a c-valuation re $* \otimes id$. Hence, Theorem 3.4 shows E is division a algebra and v_1 is a c-valuation of E with respect to $*_E$ with residue degree 4 over K.

For the residue degree 2 case we use another extension of v to K. Let v_2 be an extension of v to K such that $\Gamma_{K,v_2} = \Gamma_F \times \mathbb{Z}$, $v_2(x) = (0, 0)$, $v_2(y) = (0, 1)$, $v_2(c) = (v(c), 0)$ for $c \in F^{\cdot}$, and $\bar{K}_{v_2} = \bar{F}(\bar{x})$ with $\bar{x}$ purely transcendental over $\bar{F}$. (In terms of places, the place associated to v_2 is the composition of the place $K \to \bar{F}(\bar{x}, \bar{y}) \cup \infty$ associated to v_1 with the place $\bar{F}(\bar{x}, \bar{y}) \to \bar{F}(\bar{x}) \cup \infty$ associated to the discrete valuation ring $\bar{F}[\bar{x}, \bar{y}]_{(\bar{y})}$.) Then, as in the preceding case, $D \otimes_F K$ is a division ring, v_2 extends to $D \otimes_F K$ with $\overline{D \otimes_F K_2} = \bar{D} \otimes_{\bar{F}} \bar{K}_{v_2} = \bar{K}_{v_2}$ and $\Gamma_{D \otimes_F K} = \Gamma_D \times \mathbb{Z}$, so $D \otimes_F K$ is totally ramified over K. Take any ordering of $\bar{F}$, and extend it to an ordering of $\bar{K}_{v_2}$ with $\bar{x} < 0$. Since $v_2(y) = (0, 1) \notin 2\Gamma_{K,v_2}$, Theorem 3.9 shows v_2 extends to Q with $\Gamma_{Q,v_2} = \Gamma_K \times \frac{1}{2}\mathbb{Z}$, and v_2 is a c-valuation of Q re $*_Q$ with residue degree 2 over K. We know by Theorem 3.1 that v_2 is a c-valuation of $D \otimes_F K$ re $* \otimes id$. So Theorem 3.4 applies again to show that v_2 is a c-valuation of E re $*_E$ and E has residue degree 2 over K.

To see that E is not isomorphic to a tensor product of quaternion algebras, we use yet another valuation. Let w be the valuation of K obtained by restriction of the standard valuation on $F((x))((y))$. So, $\bar{K}_w = F$ and $\Gamma_{K,w} = \mathbb{Z} \times \mathbb{Z}$, with $w(x) = (1, 0)$ and $w(y) = (0, 1)$. Because x and y map to $\mathbb{Z}/2\mathbb{Z}$-independent elements of $\Gamma_{K,w}/2\Gamma_{K,w}$, w extends to a valuation of Q totally ramified over K with $\Gamma_{Q,w} = \frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z}$ (cf. [TW, Proposition 3.5]). Now, w restricts to the trivial valuation on F, which extends to the trivial valuation on D. Morandi's product theorem, Proposition 3.3 above, applies to E viewed as $D \otimes_F Q$, showing that w on D and Q extends to a valuation on E with $\bar{E}_w \cong \bar{D}_w \otimes_{\bar{F}_w} \bar{Q}_w \cong D \otimes_F F \cong D$. Theorem 4.3 below shows that since D is not a tensor product of quaternion algebras, E cannot be either.

(2) Let $N = F(x^2) \subseteq L = F(x)$. The valuation v on F extends to a valuation (also called v) on L with $\Gamma_L = \Gamma_F \times \mathbb{Z}$, $v(x) = (0, 1)$, $\bar{L} = \bar{F}$, and $\Gamma_N = \Gamma_F \times 2\mathbb{Z}$. View A as $(D \otimes_F N) \otimes_N L$. Since $\Gamma_D \cap \Gamma_N = \Gamma_F$ and $\bar{N} = \bar{F}$, Theorem 3.4 applies with D for the E and N for the T, showing that v

extends to a valuation of the division ring $D \otimes_F N$ which is a c-valuation with respect to the involution $* \otimes id$ of the first kind with the same residue degree as D over F, and $\Gamma_{D \otimes_F N} = \Gamma_D \times 2\mathbb{Z}$. Then Lemma 3.5 applies to see that v extends to a valuation of the division algebra $(D \otimes_F N) \otimes_N L$ which is a c-valuation re the involution $(* \otimes id) \otimes *_L$ of the second kind, with the same residue degree as $D \otimes_F N$ over N. In the isomorphism $(D \otimes_F N) \otimes_N L \cong D \otimes_F L = A$, the involution $(* \otimes id) \otimes *_L$ corresponds to $*_A$.

It remains only to verify that A is not a product of quaternion algebras. For this we use the valuation w of part 1, with $\Gamma_{L,w} = \mathbb{Z}$ and $\bar{L}_w = \bar{F}$. Just as in part 1, we see that w extends to $A = D \otimes_F L$ with $\bar{A}_w = D$. Theorem 4.3 thus shows that A cannot be a product of quaternion algebras, since $\bar{A}_w$ is not such a product. ∎

To complete the proof of Proposition 4.2, we need the next theorem on decompositions into quaternion algebras In the course of the proof we will use the machinery of armatures developed in [TW, Sect. 2]. For any ring A, we write $A^{\bullet}$ for the group of units of A. Recall that if A is a finite-dimensional algebra over a field F, an *armature* of A is an abelian subgroup $\mathscr{A}$ of $A^{\bullet}/F^{\bullet}$, such that $|\mathscr{A}| = \dim_F A$ and A is generated as an F-vector space by the inverse images of elements of $\mathscr{A}$. Associated to $\mathscr{A}$ there is the armature pairing $B_{\mathscr{A}}: \mathscr{A} \times \mathscr{A} \to \mu(F)$, where $\mu(F)$ denotes the group of roots of unity of F. The pairing $B_{\mathscr{A}}$ defined by $(aF^{\bullet}, bF^{\bullet}) \mapsto aba^{-1}b^{-1}$ is $\mathbb{Z}$-bilinear and symplectic. It was shown in [TW, Lemma 2.5(iii) and Proposition 2.1] that if $B_{\mathscr{A}}$ is nondegenerate then A is central simple over F and is isomorphic to a tensor product of symbol algebras. Conversely, any tensor product of symbol algebras has an armature with nondegenerate symplectic pairing. Given any subgroup $\mathscr{K}$ of an armature $\mathscr{A}$ of an algebra A, let $F[\mathscr{K}]$ denote the F-subspace (and subalgebra) of A generated by the inverse images in $A^{\bullet}$ of the elements of $\mathscr{K}$. It is easy to check (cf. [TW, Example 2.4(c)]) that $\mathscr{K}$ is an armature of $F[\mathscr{K}]$.

THEOREM 4.3. *Let A be a division algebra finite-dimensional over its center F. Suppose A has a valuation v with* $\operatorname{char}(\bar{A}) \neq 2$. *If A is isomorphic is a tensor product of quaternion algebras, then either $\bar{A} = Z(\bar{A})$, or $\bar{A}$ is isomorphic to a tensor product of quaternion algebras over $Z(\bar{A})$.*

Proof. If $A = Q_1 \otimes_F \cdots \otimes_F Q_n$, where each Q_m is a quaternion algebra with standard generators i_m, j_m, then the subgroup $\mathscr{A}$ of $A^{\bullet}/F^{\bullet}$ generated by the images of $\{i_1, j_1, \ldots, i_n, j_n\}$ is clearly an armature of A. The associated armature pairing $B_{\mathscr{A}}$ is a nondegenerate symplectic map $\mathscr{A} \times \mathscr{A} \to \{\pm 1\}$. We will show $\bar{A}$ is a tensor product of quaternion algebras by producing an armature for it.

The valuation $v: A^{\bullet} \to \Gamma_A$ induces a group homomorphism $w: \mathscr{A} \to \Gamma_A/\Gamma_F$. Let $\mathscr{K} = \ker(w)$, and let $\mathscr{L} = \mathscr{K} \cap \mathscr{K}^{\perp}$, where $\mathscr{K}^{\perp}$ is the orthogonal set of $\mathscr{K}$ with respect to $B_{\mathscr{A}}$. Let $L = F[\mathscr{L}]$ and $E = F[\mathscr{K}]$, subalgebras of A. Note for later use that

$$|\mathscr{K}| = |\mathscr{A}|/|\mathrm{im}(w)| \geqslant [A:F]/|\Gamma_A : \Gamma_F| \geqslant [\bar{A}:\bar{F}]. \tag{4.4}$$

Because $B_{\mathscr{A}}$ is trivial on $\mathscr{L}$ the algebra L is commutative, hence a field. Indeed, it is easy to verify (cf. [TW, Lemma 2.5]) that L is the center of E.

We claim that L is unramified over F re v. For this, let $\mathscr{L}'$ be the inverse image of $\mathscr{L}$ in $A^{\bullet}$, and let $U_{\mathscr{L}} = \mathscr{L}' \cap U_A$, where $U_A = \{a \in A^{\bullet} \mid v(a) = 0\}$. Observe that $U_{\mathscr{L}} \cdot F^{\bullet} = \mathscr{L}'$ as $\mathscr{L} \subseteq \ker(w)$, and $U_{\mathscr{L}} \cap F^{\bullet} = U_F$. Hence, $U_{\mathscr{L}}/U_F = U_{\mathscr{L}}/(U_{\mathscr{L}} \cap F^{\bullet}) \cong U_{\mathscr{L}} \cdot F^{\bullet}/F^{\bullet} = \mathscr{L}$. The projection map $V_L \to \bar{L}$ induces a group homomorphism $\rho: U_{\mathscr{L}}/U_F \to \bar{L}^{\bullet}/\bar{F}^{\bullet}$. This ρ must be injective. For, if $u \in U_{\mathscr{L}} - U_F$ and $\rho(u) = 1$, then $u \notin F$ while $\bar{u} \in \bar{F}$ and $u^2 \in F$ (as $\mathscr{A}$ has exponent 2). Then, as $\mathrm{char}(\bar{F}) \neq 2$, v has two different extensions to the field $F(u)$. But because v extends to A it must extend uniquely from F to each subfield of A. So, there is no such u, and ρ is injective. Let $\bar{\mathscr{L}}$ denote the image of ρ. So $\bar{\mathscr{L}}$ is a subgroup of $\bar{L}^{\bullet}/\bar{F}^{\bullet}$ and $\bar{\mathscr{L}} \cong U_{\mathscr{L}}/U_F \cong \mathscr{L}$. Hence $\bar{\mathscr{L}}$ has exponent (at most) 2. By Kummer theory, $[\bar{L}:\bar{F}] \geqslant |\bar{\mathscr{L}}| = |\mathscr{L}| = [L:F]$. The fundamental inequality for extending valuations then shows $[\bar{L}:\bar{F}] = [L:F]$, so L is unramified over F, as claimed.

Let $\mathscr{E}$ be the image of $\mathscr{K}$ under the canonical homomorphism $E^{\bullet}/F^{\bullet} \to E^{\bullet}/L^{\bullet}$. It is shown in [TW, Proof of Corollary 2.8] that the kernel of $\mathscr{K} \to \mathscr{E}$ is $\mathscr{L}$, and $\mathscr{E}$ is an armature of E as an L-algebra, with nondegenerate armature pairing $B_{\mathscr{E}}$ corresponding to the pairing on $\mathscr{K}/\mathscr{L}$ (induced by $B_{\mathscr{A}}$) under the isomorphism $\mathscr{E} \cong \mathscr{K}/\mathscr{L}$. Let $\mathscr{E}'$ be the inverse image of $\mathscr{E}$ in $E^{\bullet}$, and let $U_{\mathscr{E}} = \mathscr{E}' \cap U_E$. Then $U_{\mathscr{E}} \cap L^{\bullet} = U_L$ and $U_{\mathscr{E}} \cdot L^{\bullet} = \mathscr{E}'$ (as $v(\mathscr{E}') \subseteq \Gamma_L$) so that $U_{\mathscr{E}}/U_L = U_{\mathscr{E}}/(U_{\mathscr{E}} \cap L^{\bullet}) \cong U_{\mathscr{E}} \cdot L^{\bullet}/L^{\bullet} = \mathscr{E}$.

The projection map $V_E \to \bar{E}$ induces a group homomorphism $\sigma: U_{\mathscr{E}}/U_L \to \bar{E}^{\bullet}/\bar{L}^{\bullet}$. The argument above for the injectivity of ρ applies just as well here to show that σ is injective. Let $\bar{\mathscr{E}}$ be the image of σ. Then $\bar{\mathscr{E}}$ is a finite abelian subgroup of $\bar{E}^{\bullet}/\bar{L}^{\bullet}$, and the commutator pairing $B_{\bar{\mathscr{E}}}: \bar{\mathscr{E}} \times \bar{\mathscr{E}} \to \mu(\bar{L})$ given by $(a\bar{L}^{\bullet}, b\bar{L}^{\bullet}) \mapsto aba^{-1}b^{-1}$ is nondegenerate since it corresponds to $B_{\mathscr{E}}$ under the isomorphism $\bar{\mathscr{E}} \cong \mathscr{E}$ (and since $\mathrm{char}(\bar{L}) \neq 2$). Let $\{e_1, ..., e_m\}$ be a set of representatives in $\bar{E}^{\bullet}$ for the elements of $\bar{\mathscr{E}}$. The nondegeneracy of $B_{\bar{\mathscr{E}}}$ implies that $e_1, ..., e_m$ are linearly independent over $\bar{L}$. (For, otherwise we can choose a dependence relation $\sum_i c_i e_i = 0$ with $c_i \in \bar{L}$ and a minimal number of nonzero c_i. Say $c_j \neq 0$ and $c_k \neq 0$, $j \neq k$. The nondegeneracy of $B_{\bar{\mathscr{E}}}$ assures that there is a $b\bar{L}^{\bullet} \in \bar{\mathscr{E}}$ with $be_jb^{-1}e_j^{-1} \neq be_kb^{-1}e_k^{-1} \in \mu(\bar{L})$. Then,

$$0 = be_j b^{-1} e_j^{-1} \sum_i c_i e_i - b\left(\sum_i c_i e_i\right) b^{-1}$$
$$= \sum_i c_i (be_j b^{-1} e_j^{-1} - be_i b^{-1} e_i^{-1})\, e_i.$$

This is a nontrivial dependence relation with fewer nonzero coefficients, a contradiction.) So, $|\bar{\mathscr{E}}| \leqslant [\bar{E}:\bar{L}]$. But since $|\mathscr{K}| \geqslant [\bar{A}:\bar{F}]$ by (4.4), and $|\mathscr{L}| = [L:F] = [\bar{L}:\bar{F}]$, we have

$$[\bar{E}:\bar{L}] \geqslant |\bar{\mathscr{E}}| = |\mathscr{E}| = |\mathscr{K}|/|\mathscr{L}| \geqslant [\bar{A}:\bar{F}]/[L:K]$$
$$= [\bar{A}:\bar{F}]/[\bar{L}:\bar{F}] = [\bar{A}:\bar{L}] \geqslant [\bar{E}:\bar{L}].$$

Hence, equality must hold here, showing that $\bar{E} = \bar{A}$ and $\{e_1, ..., e_n\}$ is actually a base of $\bar{A}$ over $\bar{L}$. This shows that $\bar{\mathscr{E}}$ is an armature of $\bar{A}$. Because the armature pairing is nondegenerate, by [TW, Proposition 2.7], $\bar{A}$ is isomorphic to a tensor product of symbol algebras of degree dividing the exponent of $\bar{\mathscr{E}}$. Since $\bar{\mathscr{E}}$ has exponent 2, these symbol algebras must all be quaternion algebras, as desired. ∎

Remark. By slight modifications of this argument one can show that if A is a valued division algebra which is a tensor product of symbol algebras (none of which has degree a multiple of char$(\bar{A})$), then $\bar{A}$ is a tensor product of symbol algebras.

One might still ask whether a c-valued division ring which is expressible as a product of quaternion algebras must, in fact, be expressible as a tensor product of $*$-closed quaternion subalgebras. Our final example shows that this need not be the case.

THEOREM 4.5. *There is a division algebra D with center F such that $D \cong Q_1 \otimes_F Q_2$, where the Q_i are quaternion algebras, but D has a c-valuation with respect to some involution $*$ of the first kind, such that no quaternion subalgebra of D is $*$-closed.*

Proof. Amitsur, Rowen, and Tignol gave in [ART, Theorem 5.2] a construction of a D with the required properties except having a c-valuation. We give a slight variation of their example which will in addition have a c-valuation.

Let $k = \mathbb{Q}(x, y)$, where x and y are algebraically independent over $\mathbb{Q}$. Let $L = k(\sqrt{x})$ and $K = k(\sqrt{x}, \sqrt{y})$, and let $N_{K/L}: K \to L$ be the norm map. To start the construction, we need a $b \in L$ with $b \in kN_{K/L}(c)$ for some $c \in K$, but $b \notin kL^2$. For this, take $b = N_{K/L}(\sqrt{x} + \sqrt{y} + 1) = x + 2\sqrt{x} + 1 - y$. Let σ be the nonidentity k-automorphism of L. If $b = ad^2$ with $a \in k$, $d \in L$, then $\sigma(b) = a\sigma(d)^2$, so $b/\sigma(b) \in L^2$. This cannot occur because $b =$

$x+2\sqrt{x+1}-y$ and $\sigma(b)=x-2\sqrt{x+1}-y$ are nonassociate irreducibles of the factorial ring $\mathbb{Q}(y)[\sqrt{x}]$ with quotient field L. So, $b\notin kL^2$.

Beginning with the biquadratic extension K/k and element b as just described, it is shown in [ART, Proof of Theorem 3.6] how to construct a generic abelian crossed product (K, U, τ) with involution of the first kind which has no $*$-closed quaternion algebras. Let $D=(K, U, \tau)$. Then, it is known that the center F of D has the form $F=k(s, t)$ with s and t algebraically independent over k; further, the restriction to F of the standard valuation v of $k((s))((t))$ extends to a valuation of D with $\bar{D}=K$. (This valuation information about D is sketched in [JW_2, Remark 5.16] and described somewhat more fully in [T_2].) The valuation we want is a refinement of v. Let u be the valuation of k obtained by restricting the standard valuation of $\mathbb{Q}((x))((y))$. Clearly u has a unique extension to $K=k(\sqrt{x}, \sqrt{y})$ and (K, u) is totally ramified over (k, u). Let V_K be the valuation ring of u on K, and let V_D be the valuation ring of v on D. Let $\pi\colon V_D\to\bar{D}=K$ be the canonical epimorphism, and let $W=\pi^{-1}(V_K)$. Then, for each $d\in D^{\cdot}$, $d\in W$ or $d^{-1}\in W$. Furthermore, $dWd^{-1}=W$ since V_D is invariant under inner automorphisms and V_K is the unique extension of $V_K\cap k$ to K. Hence, W is a valuation ring of D; let w be the associated valuation. Then $\bar{D}_w=\overline{V_K}=\mathbb{Q}$. So, as $\bar{F}_w\subseteq\bar{D}_w$, $\bar{F}_w=\mathbb{Q}$. Therefore, by the inertial equality (1.3), D is totally ramified over F re w. Thus, Theorem 3.1 shows that w is a c-valuation of D re $*$, as desired. ∎

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